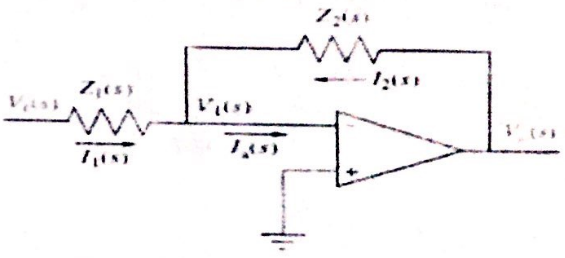
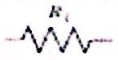
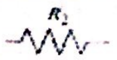
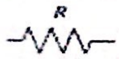
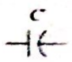
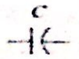
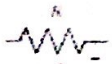
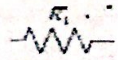
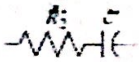
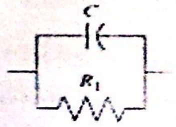
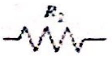
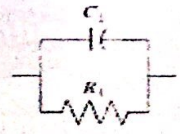
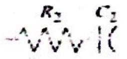
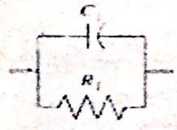
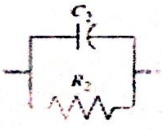
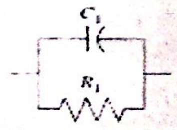
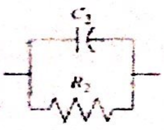
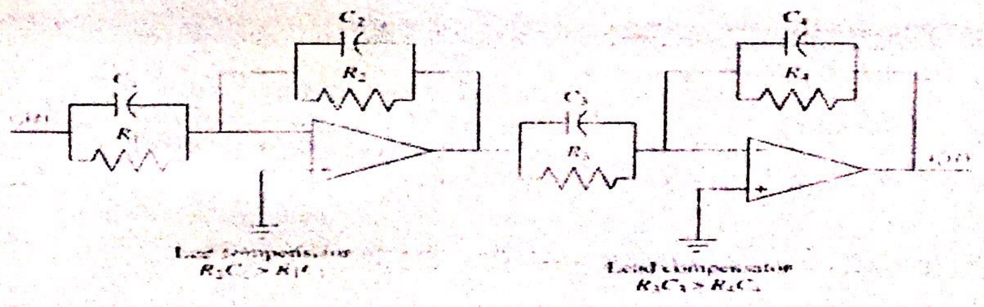
Design Via Root Locus

Active-Circuit Realization



$$\boxed{\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)}}$$

Function	$Z_1(s)$	$Z_2(s)$	$G_c(s) = -\frac{Z_2(s)}{Z_1(s)}$
Gain	 R_1	 R_2	$-\frac{R_2}{R_1}$
Integration	 R	 C	$-\frac{\frac{1}{RC}}{s}$
Differentiation	 C	 R	$-RCs$
PI controller	 R_1	 R_2 C	$-\frac{R_2}{R_1}\frac{\left(s+\frac{1}{R_2C}\right)}{s}$
PD controller	 C R_1	 R_2	$-R_2C\left(s+\frac{1}{R_1C}\right)$
PID controller	 C_1 R_1	 R_2 C_2	$-\left[\left(\frac{R_2}{R_1}+\frac{C_1}{C_2}\right)+R_2C_1s+\frac{\frac{1}{R_1C_2}}{s}\right]$
Lag compensation	 C_1 R_1	 C_2 R_2	$-\frac{C_1}{C_2}\frac{\left(s+\frac{1}{R_1C_1}\right)}{\left(s+\frac{1}{R_2C_2}\right)}$ where $R_2C_2 > R_1C_1$
Lead compensation	 C_1 R_1	 C_2 R_2	$-\frac{C_1}{C_2}\frac{\left(s+\frac{1}{R_1C_1}\right)}{\left(s+\frac{1}{R_2C_2}\right)}$ where $R_1C_1 > R_2C_2$



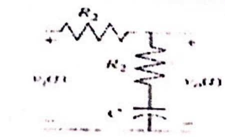
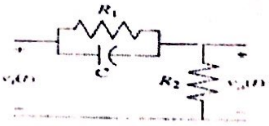
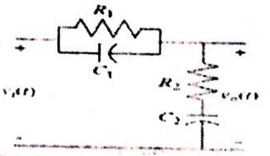
Lag compensator
$R_2C_2 > R_1C_1$

Lead compensator
$R_3C_3 > R_4C_4$

Lag-lead compensator implemented with operational amplifiers

الورقة 12

Passive-Circuit Realization

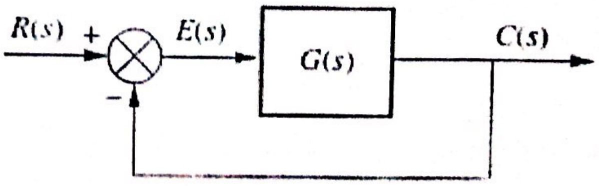
Function	Network	Transfer function, $\frac{V_o(s)}{V_i(s)}$
Lag compensation	 R_2 R_2 C $v_i(t)$ $v_o(t)$ + + − −	$\frac{R_2}{R_1+R_2}\frac{s+\frac{1}{R_2C}}{s+\frac{1}{(R_1+R_2)C}}$
Lead compensation		$\frac{s+\frac{1}{R_1C}}{s+\frac{1}{R_1C}+\frac{1}{R_2C}}$
Lag-lead compensation	 R_1 C_1 R_2 C_2 $v_i(t)$ $v_o(t)$ + +	$\frac{\left(s+\frac{1}{R_1C_1}\right)\left(s+\frac{1}{R_2C_2}\right)}{s^2+\left(\frac{1}{R_1C_1}+\frac{1}{R_2C_2}+\frac{1}{R_2C_1}\right)s+\frac{1}{R_1R_2C_1C_2}}$

Function	Compensator	Transfer function	Characteristics
Improve steady-state error	PI	$K\frac{s+z_c}{s}$	1. Increases system type. 2. Error becomes zero. 3. Zero at $-z_c$ is small and negative. 4. Active circuits are required to implement.
Improve steady-state error	Lag	$K\frac{s+z_c}{s+p_c}$	1. Error is improved but not driven to zero. 2. Pole at $-p_c$ is small and negative. 3. Zero at $-z_c$ is close to, and to the left of, the pole at $-p_c$. 4. Active circuits are not required to implement.
Improve transient response	PD	$K(s+z_c)$	1. Zero at $-z_c$ is selected to put design point on root locus. 2. Active circuits are required to implement. 3. Can cause noise and saturation; implement with rate feedback or with a pole (lead).
Improve transient response	Lead	$K\frac{s+z_c}{s+p_c}$	1. Zero at $-z_c$ and pole at $-p_c$ are selected to put design point on root locus. 2. Pole at $-p_c$ is more negative than zero at $-z_c$. 3. Active circuits are not required to implement.
Improve steady-state error and transient response	PID	K [illegible] $/s$	[illegible] at $-z_{\text{lag}}$ and pole at origin improve steady state error. 2. Lead zero at $-z_{\text{lead}}$ improves transient response. 3. Lag zero at $-z_{\text{lag}}$ is close to, and to the left of, the origin 4. Lead zero at $-z_{\text{lead}}$ is selected to put design point on root locus. 5. Active circuits required to implement. 6. Can cause noise and saturation; implement with rate feedback or with an additional pole.
Improve steady-state error and transient response	Lag-lead	$K\frac{(s+z_{\text{lag}})(s+z_{\text{lead}})}{(s+p_{\text{lag}})(s+p_{\text{lead}})}$	1. Lag pole at $-p_{\text{lag}}$ and lag zero at $-z_{\text{lag}}$ are used to improve steady-state error. 2. Lead pole at $-p_{\text{lead}}$ and lead zero at $-z_{\text{lead}}$ are used to improve transient response. 3. Lag pole at $-p_{\text{lag}}$ is small and negative. 4. Lag zero at $-z_{\text{lag}}$ is close to, and to the left of, lag pole at $-p_{\text{lag}}$. 5. Lead zero at $-z_{\text{lead}}$ and lead pole at $-p_{\text{lead}}$ are selected to put design point on root locus. 6. Lead pole at $-p_{\text{lead}}$ is more negative than lead zero at $-z_{\text{lead}}$. 7. Active circuits are not required to implement.

Example

Design a PI controller to drive the step response error to zero for the unity feedback system shown in Figure below, where

$$G(s) = \frac{K}{(s+1)(s+3)(s+10)}$$

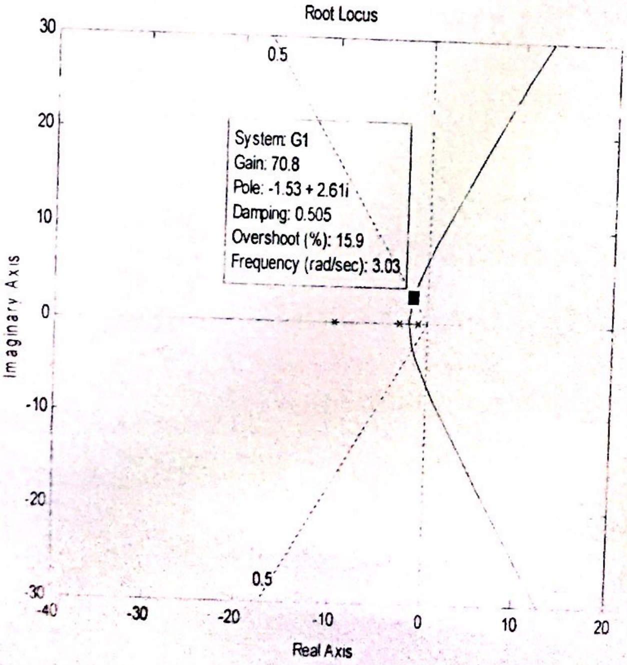


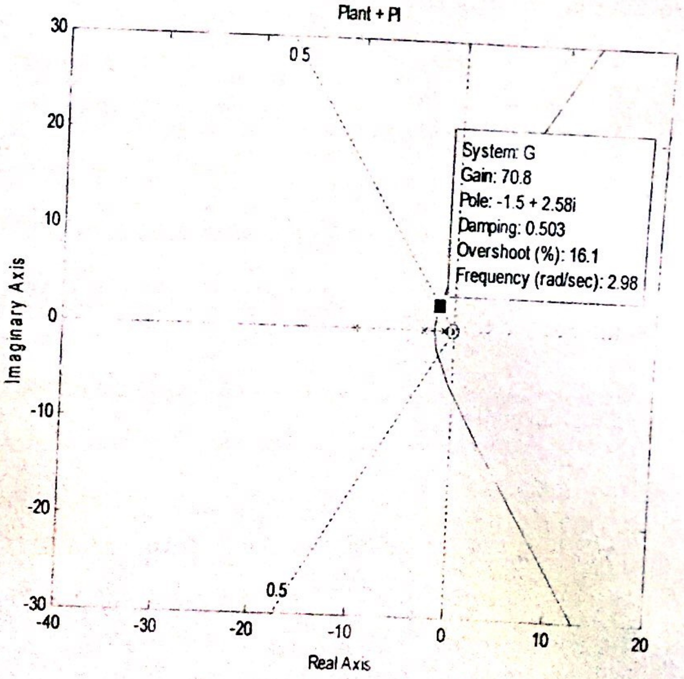
The system operates with a damping ratio of 0.5. Compare the specifications of the uncompensated and compensated systems.

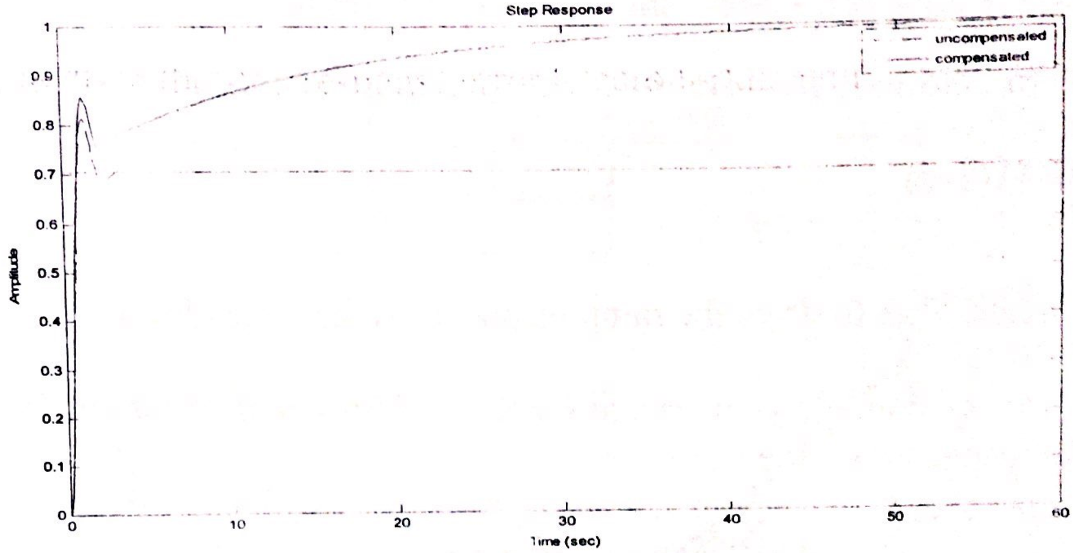
Uncompensated system: Search along the $\zeta = 0.5$ line and find the operating point is at $-1.5356 \pm j2.6598$ with K = 73.09. Hence, $\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \text{x}100 = 16.3\%$; $T_s = \dfrac{4}{1.5356} = 2.6$ seconds; $K_p = \dfrac{73.09}{30} = 2.44$. A higher-order pole is located at -10.9285.

Compensated: Add a pole at the origin and a zero at -0.1 to form a PI controller. Search along the $\zeta = 0.5$ line and find the operating point is at $-1.5072 \pm j2.6106$ with K = 72.23. Hence, the estimated performance specifications for the compensated system are: $\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \text{x}100 = 16.3\%$; $T_s = \dfrac{4}{1.5072} = 2.65$ seconds; $K_p = \infty$. Higher-order poles are located at -0.0728 and -10.9125. The compensated system should be simulated to ensure effective pole/zero cancellation.

```
close all
K=1
G1=tf(K,poly([-1 -3 -10]))      %G1=K/(s+1)(s+3)(s+10)
rlocus(G1)
sgrid(0.5,0)
Gc=tf([1 0.1],[1 0])      %Gc=(s+0.01)/s
G=G1*Gc
figure
rlocus(G)
title('Plant + PI')
sgrid(0.5,0)
figure
K=72
T=feedback(K*G1,1)
T1=feedback(K*G,1)
figure
step(T,T1)
legend('uncompensated','compensated')
```







Dominate poles $S_{1,2}$= -1.5±2.58j

$$G_p = \frac{K}{(s+1)(s+3)(s+10)} = \frac{K}{s^3+14s^2+43s+30}$$

$$K = \left.\frac{-1}{G_p}\right]_{s=-1.5+2.6j} = 70.5$$

For unity feedback 1+KG$_p$(s)H(s) = 0

$$s^3+14s^2+43s+30+K = s^3+14s^2+43s+100.5$$

Roots([1 14 43 100.5]) ….. -1.5±$2.6j$, -10.9

Plant with PI

$$G = G_cG_p = \frac{(s+0.1)}{s} \cdot \frac{K}{(s+1)(s+3)(s+10)}$$

$$K = -\left.\frac{s(s+1)(s+3)(s+10)}{(s+0.1)}\right]_{s=-1.5+2.58j} = 71.8$$

For unity feedback 1+KG(s)H(s) = 0

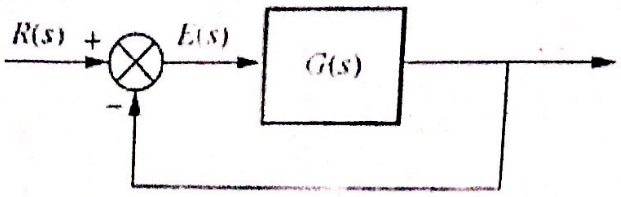
$$s(s+1)(s+3)(s+10)+K(s+0.1) = s^4+14s^3+43s^2+30s+71.8s+7.18 = 0$$

Roots([1 14 43 102 7.18]) ….. -1.5±$2.6j$, - 0.0728 , -10.9

Example

Consider the unity feedback system shown in Figure below,

$$G(s) = \frac{K}{s\,(S+2)\,(S+5)}$$



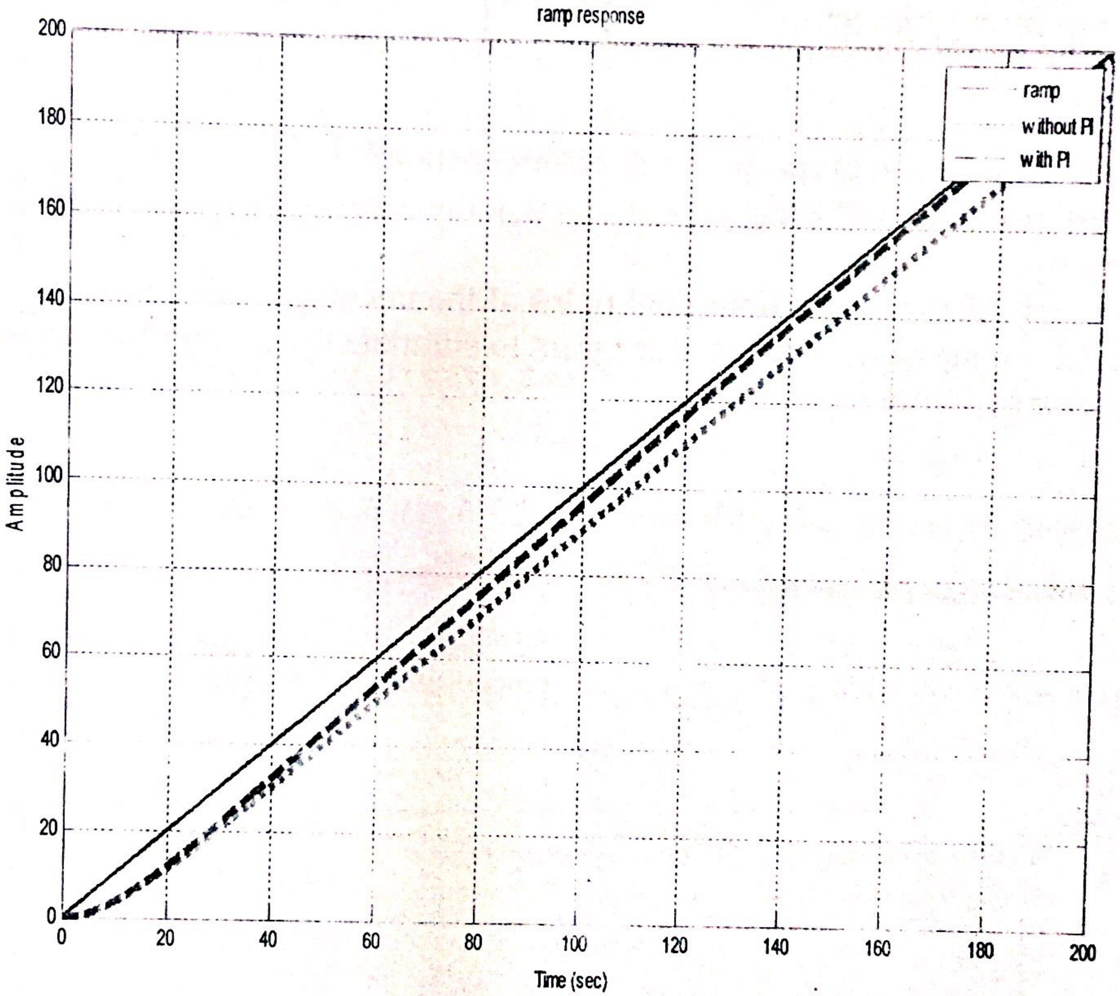
a. Design a PI controller to drive the ramp response error to zero for any K that yields stability.

b. Use MATLAB to simulate your design for K = 1. Show both the input ramp and the output responses on the same plot.

……

a. Insert a cascade compensator, such as $G_c(s) = \dfrac{s+0.01}{s}$

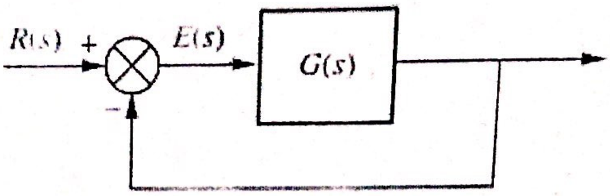
```
close all
K=1
G1=zpk([],[0,-2,-5],K)   %G1=1/s(s+2)(s+5)
Gc=zpk([-0.01],[0],1)    %Gc=(s+0.01)/s
G=G1*Gc
rlocus(G)
Tp= feedback(G1,1)
Tpc=feedback(G,1)
T1=tf(1,[1,0])           %Form 1/s to integrate step input
Tp=Tp*T1
Tpc=Tpc*T1
t=0:0.1:200;
step(T1,Tp,Tpc,t)            %Show input ramp and ramp
response
legend('ramp','without PI','with PI')
title('ramp response')
```


ramp response
Amplitude
Time (sec)
ramp
without PI
with PI
0
20
40
60
80
100
120
140
160
180
200
0
20
40
60
80
100
120
140
160
180
200

Example

The unity feedback system shown in Figure below is operating with 10% overshoot.

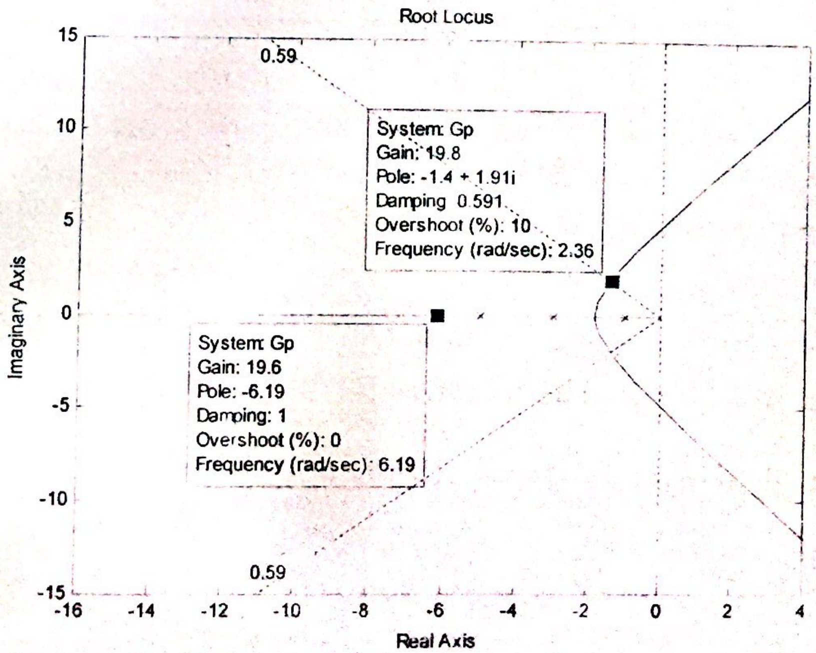
$$G(s) = \frac{K}{(s+1)(s+3)(s+5)}$$



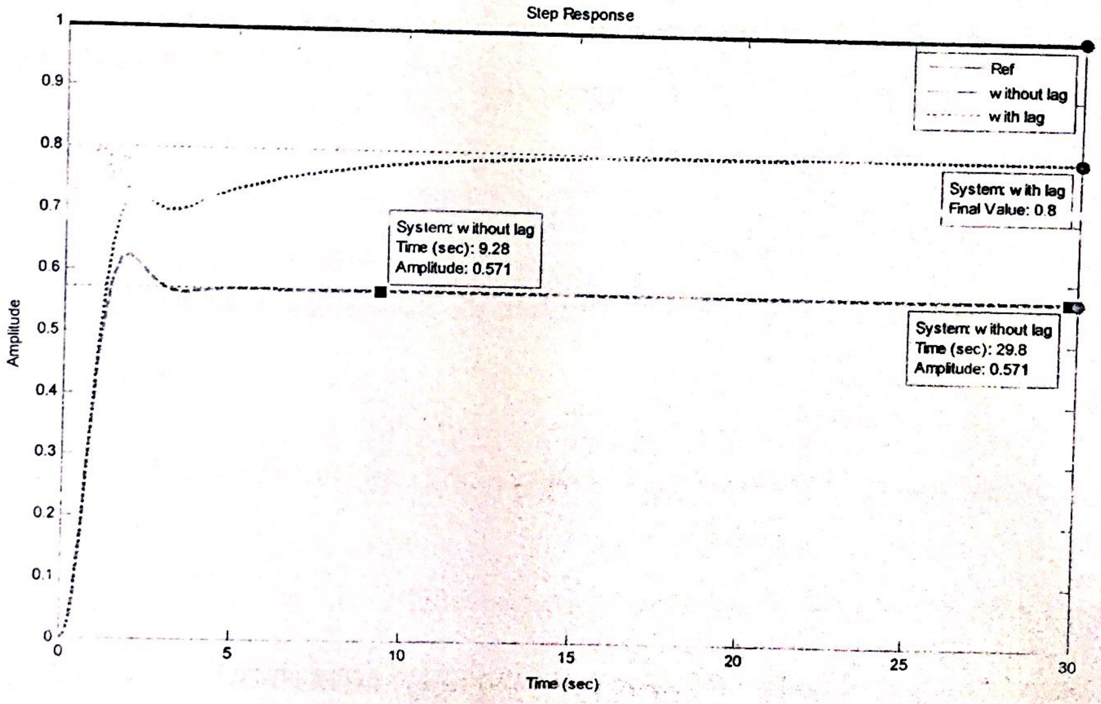
a. What is the value of the appropriate static error constant?
b. Find the transfer function of a lag network so that the appropriate static error constant equals 4
without appreciably changing the dominant poles of the uncompensated system.
c. Use MATLAB or any other computer program to simulate the system to see the effect of your compensator.

a. Searching along the 126.16° line (10% overshoot, $\zeta = 0.59$), find the operating point at $-1.4 + j1.92$ with K = 20. Hence, $K_p = \frac{20}{1 \times 5 \times 3} = 1.333$.

b. A 3x improvement will yield $K_p = 4$. Use a lag compensator, $G_c(s) = \frac{s+0.3}{s+0.1}$.



```
close all
K=1
Gp=zpk([],[-1,-3,-5],K)  %G1=1/s(s+2)(s+5)
rlocus(Gp)
sgrid(0.59,0)
figure
Gc=zpk([-0.3],[-0.1],1)
G=Gp*Gc
rlocus(Gp)
figure
K=20                      % from rootlocus os=10%
Tp= feedback(K*Gp,1)
Tpc=feedback(K*G,1)
t=0:0.01:20
T=1
step(Tp/Tp,Tp,Tpc)
legend('Ref','without lag','with lag')
```



PROBLEM: A unity feedback system with the forward transfer function

$$G(s) = \frac{K}{s(s+7)}$$

is operating with a closed-loop step response that has 15% overshoot. Do the following:

a. Evaluate the steady-state error for a unit ramp input.

b. Design a lag compensator to improve the steady-state error by a factor of 20.

c. Evaluate the steady-state error for a unit ramp input to your compensated system.

d. Evaluate how much improvement in steady-state error was realized.

a. Searching along the 15% overshoot line, we find the point on the root locus at –3.5 + j5.8 at a gain of $K = 45.84$. Thus, for the uncompensated system, $K_v = \lim_{s \to 0} sG(s) = K/7 = 45.84/7 = 6.55$.

Hence, $e_{ramp\_uncompensated}(\infty) = 1/K_v = 0.1527$.

b. Compensator zero should be 20x further to the left than the compensator pole.

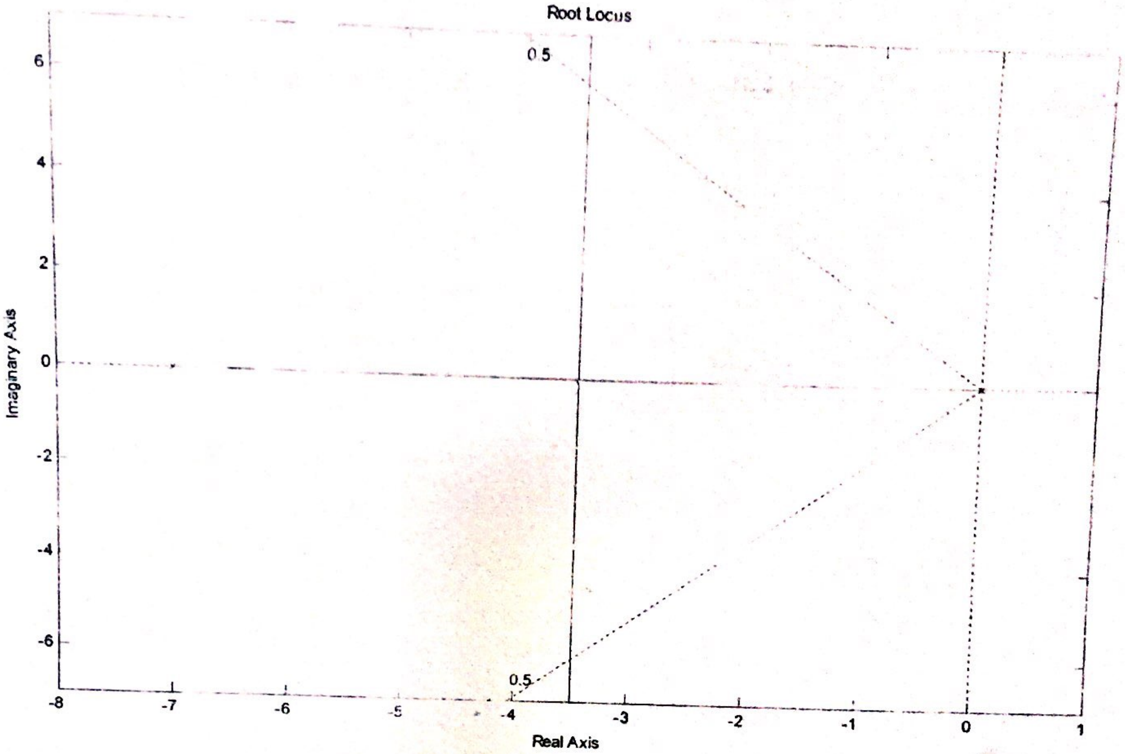
Arbitrarily select $G_c(s) = \dfrac{(s+0.2)}{(s+0.01)}$.

c. Insert compensator and search along the 15% overshoot line and find the root locus at

–3.4 + j5.63 with a gain, $K = 44.64$. Thus, for the compensated

system, $K_v = \dfrac{44.64(0.2)}{(7)(0.01)} = 127.5$ and $e_{ramp\_compensated}(\infty) = \dfrac{1}{K_v} = 0.0078$.

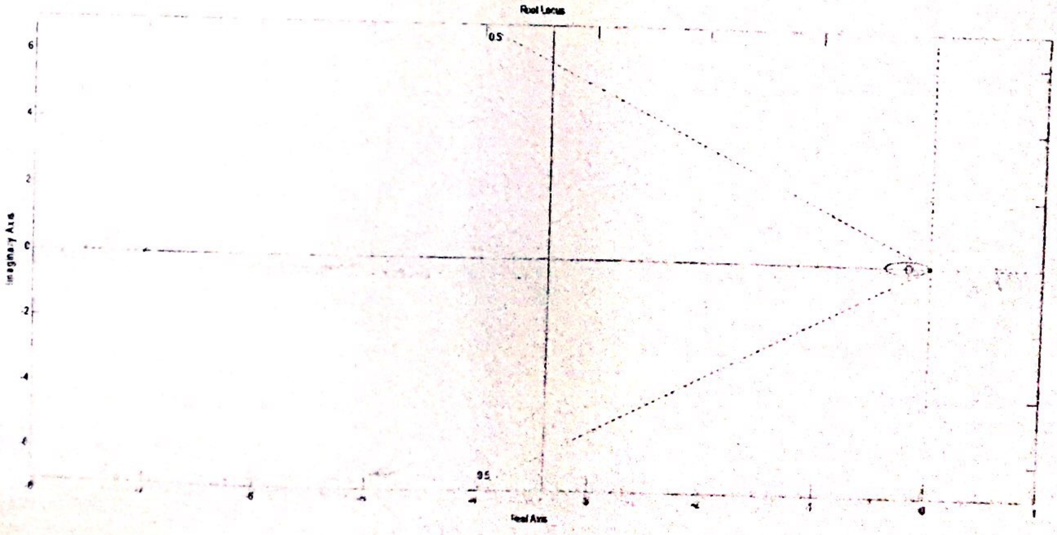
d. $\dfrac{e_{ramp\_uncompensated}}{e_{ramp\_compensated}} = \dfrac{0.1527}{0.0078} = 19.58$



Dominate poles $S_{1,2}$= -3.5±5.8j

$$G_p = \frac{K}{s(s+7)} = \frac{K}{s^2+7s}$$

$$K = \left.\frac{-1}{G_p}\right]_{s=-3.5+5.8j} = 45.89$$



```
close all
K=1
Gp=zpk([],[0,-7],K)   %G1=1/s(s+2)(s+5)
rlocus(Gp)
sgrid(0.5,0)
figure
Gc=zpk([-0.2],[-0.01],1)
G=Gp*Gc
rlocus(G)
figure
K=45.85                       % from rootlocus os=15%
Tp= feedback(K*Gp,1)
Tpc=feedback(K*G,1)
K=72
T1=tf(1,[1,0])                %Form 1/s to integrate step input
Tp=Tp*T1
Tpc=Tpc*T1
t=0:0.1:200;
step(T1,Tp,Tpc)                %Show input ramp and ramp response
legend('ramp','without lag','with lag')
title('ramp response')
```